

SPECTRAL CHARACTERISTICS AND STABLE RANKS FOR THE SARASON ALGEBRA $H^\infty + C$

RAYMOND MORTINI AND BRETT D. WICK*

INTRODUCTION

We prove a Corona type theorem with bounds for the Sarason algebra $H^\infty + C$ and determine its spectral characteristics, thus continuing a line of research initiated by N. Nikolski. We also determine the Bass, the dense, and the topological stable ranks of $H^\infty + C$.

To fix our setting, let A be a commutative unital Banach algebra with unit e and $M(A)$ its maximal ideal space. The following concept of spectral characteristics was introduced by N. Nikolski [15]. For $a \in A$, let \hat{a} denote the Gelfand transform of a . We let

$$\delta(a) = \min_{t \in M(A)} |\hat{a}(t)|.$$

Note that $\delta(a) \leq \|\hat{a}\|_\infty \leq \|a\|_A$. When $a = (a_1, \dots, a_n) \in A^n$ we define

$$\delta_n(a) = \min_{t \in M(A)} |\hat{a}(t)|,$$

where $|\hat{a}(t)| = \sum_{j=1}^n |\hat{a}_j(t)|$ for $t \in M(A)$ and we let

$$\|a\|_{A^n} = \max\{\|a_1\|_A, \dots, \|a_n\|_A\}.$$

Typically, one defines $|\hat{a}(t)| = |\hat{a}(t)|_2 := (\sum_{j=1}^n |\hat{a}_j(t)|^2)^{1/2}$ and $\|a\|_{A^n} = \|a\|_2 := (\sum_{j=1}^n \|a_j\|_A^2)^{1/2}$. Our later calculations will be easier though with the present definition.

Let δ be a real number satisfying $0 < \delta \leq 1$. We are interested in finding, or bounding, the functions

$$c_1(\delta, A) = \sup\{\|a^{-1}\|_A : \|a\|_A \leq 1, \delta(a) \geq \delta\}$$

and

$$(0.1) \quad c_n(\delta, A) = \sup\left\{\inf\{\|b\|_{A^n} : \sum_{j=1}^n a_j b_j = e\}, \|a\|_{A^n} \leq 1, \delta_n(a) \geq \delta\right\},$$

when A is the Sarason algebra $H^\infty + C$. If a is not invertible, we define $\|a^{-1}\| = \infty$.

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It should be clear that $1 \leq c_n(\delta, A) \leq c_{n+1}(\delta, A)$ and, if $0 < \delta' \leq \delta \leq 1$, then $c_n(\delta, A) \leq c_n(\delta', A)$. This implies the existence of a *critical constant*, denoted here by $\delta_n(A)$ such that

$$c_n(\delta, A) = \infty \text{ for } 0 < \delta < \delta_n(A) \text{ and } c_n(\delta, A) < \infty \text{ for } \delta_n(A) < \delta \leq 1.$$

It is clear that if A is a uniform algebra, then $\delta_1(A) = 0$ and $c_1(\delta, A) = 1/\delta$. It is not known for which uniform algebras $\delta_n(A) > 0$.

If $A = H^\infty$, the algebra of bounded holomorphic functions in the unit disk \mathbb{D} , then the famous Carleson Corona Theorem tells us that $\delta_n(H^\infty) = 0$ for each n . Estimates for $c_n(\delta, H^\infty)$ were given, among others, by Nikolski [14], Rosenblum [21], and Tolokonnikov [25]. The best known estimate today seems to appear in [27] and [28], see also [29]. Here, for the lower bound, δ is close to 0, and κ is a universal constant:

$$\kappa \delta^{-2} \log \log(1/\delta) \leq c_n^{(2)}(\delta, H^\infty) \leq \frac{1}{\delta} + \frac{17}{\delta^2} \log \frac{1}{\delta},$$

where $c_n^{(2)}(\delta, A)$ denotes the spectral characteristic $c_n(\delta, A)$ above (see (1.2)), whenever defined with the euclidian norms $|\cdot|_2$ and $\|\cdot\|_2$.

The structure of the paper is as follows. In Section 1 we consider the problem of solving Bezout equations in the Sarason algebra $H^\infty + C$. Indeed, we prove that the Corona Theorem with bounds holds in $H^\infty + C$ (that is $\delta_n(H^\infty + C) = 0$ for each n). We also give explicit estimates of the associated spectral characteristics.

In Section 2 we present different notions of stable ranks that are relevant to the topic of this paper. We also show the relationships between these various notions of stable ranks.

In Section 3 we determine the Bass and topological stable ranks of $H^\infty + C$. These results can be considered as a generalization of the Corona Theorem for H^∞ . In particular, we investigate whether any Bezout equation $af + bg = 1$ in $H^\infty + C$ admits a solution where a itself is invertible. We also show that on $H^\infty + C$ -convex sets in $M(H^\infty + C)$ zero free functions admit $H^\infty + C$ -invertible approximants. This will be used to determine the dense stable rank of $H^\infty + C$.

1. SPECTRAL CHARACTERISTICS FOR THE SARASON ALGEBRA $H^\infty + C$

Since we are dealing only with uniform algebras A , we shall identify the elements in A with their Gelfand transform \hat{f} .

Let $L^\infty(\mathbb{T})$ denote the algebra of essentially bounded, Lebesgue measurable functions on the unit circle \mathbb{T} . Then a *Douglas algebra* is a uniformly closed subalgebra of $L^\infty(\mathbb{T})$ that properly contains the algebra, H^∞ , of (boundary values) of bounded analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We refer the reader to the book of J. B. Garnett [7] for items and results not explicitly defined here.

The simplest example of a Douglas algebra is the algebra $H^\infty + C$ of sums of (boundary values) of bounded analytic functions and complex-valued continuous functions on \mathbb{T} . This algebra is frequently called the Sarason algebra since it was first shown by Sarason, that this space is a closed subalgebra of $L^\infty(\mathbb{T})$, see [22]. He showed that (on \mathbb{T}), $H^\infty + C$ is the uniform closure of the set of functions $\{f\bar{z}^n : f \in H^\infty, n \in \mathbb{N}\}$. Thus, $H^\infty + C = [H^\infty, \bar{z}]$, the closed algebra generated by H^∞ and the monomials \bar{z}^n .

Let $M(H^\infty)$ denote the maximal ideal space of H^∞ . It is well known that the spectrum of $M(H^\infty + C)$ can be identified with the Corona of H^∞ , namely the set $M(H^\infty) \setminus \mathbb{D}$, see [7, p. 377]. We denote by

$$Z(f) = \{m \in M(H^\infty) : f(m) = 0\},$$

the zero set of a function in H^∞ . We will also need the notion of pseudo-hyperbolic distance, $\rho(x, m)$, of two points $m, x \in M(H^\infty)$. Recall that

$$\rho(x, m) = \inf\{|f(m)| : \|f\|_\infty \leq 1, f(x) = 0\},$$

and that for $z, w \in \mathbb{D}$ we obtain $\rho(z, w) = \left| \frac{z-w}{1-\bar{w}z} \right|$, where we identify the point evaluation functional $f \mapsto f(z)$ with the point z itself.

Also, for a set E in $M(H^\infty)$, let $\rho(E, x) = \inf\{\rho(e, x) : e \in E\}$ be the pseudo-hyperbolic distance of E to a point $x \in M(H^\infty)$. Similarly, $\rho(E, U) = \inf\{\rho(E, u) : u \in U\} = \inf\{\rho(e, u) : e \in E, u \in U\}$. It is well known that for closed sets E the distance functions $\rho(\cdot, \cdot)$ and $\rho(\cdot, E)$ are lower semi-continuous on $M(H^\infty)$, see [8] and [9]. In particular, if $\rho(E, x) > \eta > 0$, then there exists an open set U containing x such that $\rho(E, U) > \eta$.

Finally, for a point $m \in M(H^\infty)$, let $P(m) = \{x \in M(H^\infty) : \rho(x, m) < 1\}$ denote the Gleason part associated with m . For example \mathbb{D} itself is the Gleason part associated with the origin. By Hoffman's theory, there exists a map L_m of \mathbb{D} onto $P(m)$ such that $\hat{f} \circ L_m$ is analytic for all $f \in H^\infty$. If (a_β) is any net in \mathbb{D} that converges to m , then L_m is given by $L_m(z) = \lim \frac{a_\beta + z}{1 + \bar{a}_\beta z}$, where the limit is taken in the topological product space $M(H^\infty)^\mathbb{D}$.

1.1. The Corona Property for $H^\infty + C$. Our first theorem is based on Axler's result that any function $u \in L^\infty$ can be multiplied by a Blaschke product into $H^\infty + C$. See [1]. Using this result it is possible to prove the following assertion.

Theorem 1.1. *The Corona Theorem with bounds holds on $H^\infty + C$.*

Proof. Step 1: We first consider H^∞ -Corona data on $M(H^\infty + C)$. The goal is to find $H^\infty + C$ solutions.

Let $f_1, \dots, f_n \in H^\infty$ satisfy $\|f_j\|_\infty \leq 1$ and $|f_1| + \dots + |f_n| \geq \delta > 0$ on $M(H^\infty + C)$. In particular, $|f_1| + \dots + |f_n| \geq \delta > 0$ a.e. on \mathbb{T} . Hence,

$$1 = \sum_{j=1}^n f_j \frac{\overline{f_j}}{\sum_{k=1}^n |f_k|^2}, \text{ a.e. on } \mathbb{T}.$$

Note that the functions $\frac{\overline{f_j}}{\sum_k |f_k|^2}$ belong to $L^\infty(\mathbb{T})$. By the Axler Multiplier Theorem, [1], there exists a Blaschke product B such that for all $j \in \{1, \dots, n\}$

$$\Phi_j := B \frac{\overline{f_j}}{\sum_k |f_k|^2} \in H^\infty + C.$$

So, a.e. on \mathbb{T} , we have $B = \sum_j \Phi_j f_j$. Moreover, $\|\Phi_j\|_\infty \leq n/\delta^2$.

By continuity, there exists an annulus $A_r := \{r \leq |z| < 1\}$ such that $|f_1| + \dots + |f_n| \geq \delta/2$ on A_r . Choose a tail B_1 of B such that $|B_1| \geq \delta/2$ on $\{|z| \leq r\}$. Let $b_1 = B/B_1$. Note that \bar{b}_1 , viewed as a function on \mathbb{T} , belongs to $H^\infty + C$.

Now consider the ideal $I(f_1, \dots, f_n, B_1)$ in H^∞ . We obviously have that $|f_1| + \dots + |f_n| + |B_1| \geq \delta/2$ on \mathbb{D} . Hence, by the H^∞ -Corona Theorem, there exists a constant depending on δ , $C(\delta)$, and functions $x_1, \dots, x_n, t \in H^\infty$ with

$$\|x_1\|_\infty + \dots + \|x_n\|_\infty + \|t\|_\infty \leq C(\delta)$$

such that $1 = \sum_{j=1}^n x_j f_j + t B_1$ on \mathbb{D} , and thus almost everywhere on \mathbb{T} we have $1 = \sum_{j=1}^n x_j f_j + t B_1$. Switching again to $H^\infty + C$, we see that a.e. on \mathbb{T}

$$B_1 = \bar{b}_1 B = \sum_{j=1}^n (\bar{b}_1 \Phi_j) f_j.$$

Hence,

$$1 = \sum_{j=1}^n x_j f_j + t \left(\sum_{j=1}^n (\bar{b}_1 \Phi_j) f_j \right) = \sum_{j=1}^n f_j \left(x_j + t \bar{b}_1 \Phi_j \right),$$

where $\sum_{j=1}^n \|x_j + t \bar{b}_1 \Phi_j\|_\infty \leq C(\delta)(1 + n^2/\delta^2) =: \chi(\delta)$. Since $H^\infty + C$ is an algebra, the functions $\varphi_j := x_j + t \bar{b}_1 \Phi_j$ belong to $H^\infty + C$. Thus, we have found a solution with bounds to the Bezout equation $\sum_{j=1}^n \varphi_j f_j = 1$ a.e. on \mathbb{T} , or equivalently, on $M(H^\infty + C)$.

Step 2: We now look at general $H^\infty + C$ Corona data.

Let $F_1, \dots, F_n \in H^\infty + C$ satisfy $\|F_j\|_\infty \leq 1$ and $\sum_{j=1}^n |F_j| \geq \delta > 0$ on $M(H^\infty + C)$. Let $\chi(\delta)$ be the function above. We uniformly approximate F_j by functions of the form $\bar{z}^{N_j} f_j$; say

$$\sum_{j=1}^n \|F_j - \bar{z}^{N_j} f_j\|_\infty \leq \varepsilon = \varepsilon(\delta) := \min\{\delta/4, [4\chi(\delta/2)]^{-1}\},$$

where $f_j \in H^\infty$ and $\|f_j\|_\infty \leq 1$.

Then $\sum_{j=1}^n |f_j| \geq \delta/2$ on $M(H^\infty + C)$. We apply Step 1 to the functions f_j and find $H^\infty + C$ functions φ_j and a constant $\chi(\delta/2)$ that bounds the sum of the norms of these functions and such that $1 = \sum_{j=1}^n \varphi_j f_j$. Now

$$\begin{aligned} u &:= \sum_{j=1}^n z^{N_j} \varphi_j F_j = \sum_{j=1}^n z^{N_j} \varphi_j (F_j - \bar{z}^{N_j} f_j) + \sum_{j=1}^n \varphi_j f_j \\ &= \gamma + 1, \end{aligned}$$

where $\gamma := \sum_{j=1}^n z^{N_j} \varphi_j (F_j - \bar{z}^{N_j} f_j) \in H^\infty + C$ is majorized by $\varepsilon \sum_{j=1}^n \|\varphi_j\|_\infty \leq \varepsilon \chi(\delta/2) \leq 1/2$. Thus $|u| \geq 1/2$ on $M(H^\infty + C)$; hence u is invertible in $H^\infty + C$ with $\|u^{-1}\|_\infty \leq 2$. We conclude that

$$1 = \sum_{j=1}^n (u^{-1} z^{N_j} \varphi_j) F_j$$

where the coefficients are bounded by $2\chi(\delta/2)$. \square

An immediate Corollary (at least for the upper bound) is the following.

Corollary 1.2. *For every integer n we have $\delta_n(H^\infty + C) = 0$ and, for δ close to 0,*

$$\kappa \delta^{-2} \log(\log(1/\delta)) \leq c_n(\delta, H^\infty + C) \leq 2\chi\left(\frac{\delta}{2}\right),$$

where $\chi(\delta) = (1 + \frac{n^2}{\delta^2})C(\delta)$ with $C(\delta)$ the best constant in the H^∞ -Corona Problem and where κ is an absolute constant.

Proof. It remains to verify the lower estimate. Here we use a result of Treil [27, p. 484], that tells us that there exist two finite Blaschke products B_1 and B_2 satisfying $|B_1| + |B_2| \geq \delta > 0$ on \mathbb{D} such that for any solution $(g_1, g_2) \in (H^\infty)^2$ of the Bezout equation $g_1 B_1 + g_2 B_2 = 1$ we have $\|g_1\|_\infty \geq \kappa \delta^{-2} \log(\log(1/\delta))$ whenever $\delta > 0$ is close to 0. Now, of course, this does not give us an example in $H^\infty + C$, since $1 = \bar{B}_1 B_1 + 0 B_2$ is a solution with coefficients bounded by 1. We proceed to the following modification.

Let m be a thin point in $M(H^\infty + C)$, that is a point lying in the closure of a thin interpolating sequence, say $(z_n) = (1 - 1/n!)$. Since the associated Blaschke product b satisfies $(1 - |z_n|^2)|b'(z_n)| \rightarrow 1$, Schwarz's Lemma implies that $(b \circ L_m)(z) = e^{i\theta} z$ for every $z \in \mathbb{D}$. Now consider the functions

$$f_1 = B_1 \circ (e^{-i\theta} b) \text{ and } f_2 = B_2 \circ (e^{-i\theta} b).$$

Clearly $|f_1| + |f_2| \geq \delta$ on \mathbb{D} and hence, viewed as functions in $H^\infty + C$, we have $|f_1| + |f_2| \geq \delta$ on $M(H^\infty + C)$. Let $(h_1, h_2) \in (H^\infty + C)^2$ be a solution of $h_1 f_1 + h_2 f_2 = 1$ in $H^\infty + C$. Since $f_j \circ L_m = B_j$, we get that $1 = (h_1 \circ L_m) B_1 + (h_2 \circ L_m) B_2$ in \mathbb{D} . Thus, by Treil's result mentioned above,

$$\|h_1\|_\infty \geq \|h_1 \circ L_m\|_\infty \geq \kappa \delta^{-2} \log(\log(1/\delta)).$$

Thus,

$$c_n(\delta, H^\infty + C) \geq c_2(\delta, H^\infty + C) \geq \kappa \delta^{-2} \log(\log(1/\delta)).$$

□

2. SEVERAL NOTIONS OF STABLE RANKS

Let A be a commutative unital ring. An n -tuple $(a_1, \dots, a_n) \in A^n$ is said to be *invertible* (or *unimodular*) if there is a solution $(x_1, \dots, x_n) \in A^n$ of the Bezout equation $\sum_{j=1}^n a_j x_j = 1$. Of course this is equivalent to saying that the ideal generated by the a_j is the whole ring. The set of all invertible n -tuples in A is denoted by $U_n(A)$.

An $(n+1)$ -tuple $(a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(A)$ is said to be *n -reducible* (or simply *reducible*) in A if there exists $(x_1, \dots, x_n) \in A^n$ such that $(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1})$ is an invertible n -tuple in A^n . It can be shown that if every invertible n -tuple in A is reducible, then every invertible $(n+1)$ -tuple is reducible (see for example [30]). The smallest integer $n \in \{1, 2, \dots\}$, for which every invertible $(n+1)$ -tuple is reducible is called the *Bass stable rank* of A and is denoted by $\text{bsr}(A)$. This notion was introduced in K -theory.

For algebras of continuous or analytic functions, this has been studied e.g. by Corach and Laroitonda [3], Rupp [17, 18, 19, 20], Suárez [23, 24] and Vasershtein [30]. It was shown by Jones, Marshall and Wolff, see [10], that the Bass stable rank of the disk algebra $A(\mathbb{D})$ is one. Later, simpler proofs were given by Corach, Suárez [4] and Rupp [17, 18]. Treil later showed that the Bass stable rank of H^∞ is one, [26].

A notion related to the Bass stable rank is that of the topological stable rank. Let A be a commutative unital Banach algebra. The smallest integer n for which the set, $U_n(A)$, of invertible n -tuples is dense in A^n is called the *topological stable rank* of A , denoted by $\text{tsr}(A)$. This notion was introduced by Rieffel, [16], in the study of C^* -algebras.

Finally, we recall two additional notions of stable ranks. Let \mathcal{B} be the class of all commutative unital Banach algebras over a field \mathbb{K} . We will always assume that algebra homomorphisms, f , between members of \mathcal{B} are continuous and satisfy $f(1_A) = 1_B$. Also, if $f : A \rightarrow B$ is an algebra homomorphism, then \underline{f} will denote the associated map given by $\underline{f} : (a_1, \dots, a_n) \mapsto (f(a_1), \dots, f(a_n))$ from A^n to B^n .

By [6, p. 542], the *dense stable rank* $\text{dsr}(A)$ of $A \in \mathcal{B}$ is the smallest integer n such that for every $B \in \mathcal{B}$ and every algebra homomorphism $f : A \rightarrow B$ with dense image the induced map $U_n(A) \rightarrow U_n(B)$ has dense image. If there is no such n , we write $\text{dsr}(A) = \infty$. We note that if for some $n \in \mathbb{N}$, all $B \in \mathcal{B}$ and all homomorphisms $f : A \rightarrow B$ with dense image the set $\underline{f}(U_n(A))$ is dense in $U_n(B)$, then $\underline{f}(U_{n+1}(A))$ is dense in $U_{n+1}(B)$, (see [6, p. 543]).

The *surjective stable rank* $\text{ssr}(A)$ of $A \in \mathcal{B}$ is the smallest integer n such that for every $B \in \mathcal{B}$ and every surjective algebra homomorphism $f : A \rightarrow B$ the induced map of $U_n(A) \rightarrow U_n(B)$ is surjective, too. Again, if there is no

such n , then we write $\text{ssr}(A) = \infty$. Let us point out, that the assumption $\underline{f}(U_n(A)) = U_n(B)$ for some n , all $B \in \mathcal{B}$ and all surjective homomorphisms $\underline{f} : A \rightarrow B$ implies that $\underline{f}(U_{n+1}(A)) = U_{n+1}(B)$. This works similarly as the proof for the corresponding statement for denseness in [6, p. 543].

In fact, let $b := (b_1, \dots, b_{n+1}) \in U_{n+1}(B)$. Consider for $I = \overline{(b_{n+1})}$ (the closure of the principal ideal generated by b_{n+1}) the quotient algebra $\tilde{B} := B/I$ and the quotient mapping $\pi : B \rightarrow \tilde{B}$. Then

$$(b_1 + I, \dots, b_n + I) \in U_n(\tilde{B}).$$

By our hypothesis, since πf is surjective, there exists $a := (a_1, \dots, a_n) \in U_n(A)$ such that $\pi f(a_j) = \pi b_j$ for $1 \leq j \leq n$. Choose $\varepsilon > 0$ so that every perturbation $(a_1 - r_1, \dots, a_n - r_n)$ of a with $\|r_j\|_A < \varepsilon$ is in $U_n(A)$ again. Using the open mapping theorem, let $\eta = \eta(\varepsilon)$ be so that

$$\{y \in B : \|y\|_B < \eta\} \subseteq f(\{x \in A : \|x\|_A < \varepsilon\}).$$

Since $f(a_j) - b_j \in I$, there exist $k_j \in B$ such that $\|f(a_j) - b_j - k_j b_{n+1}\|_B < \eta$ ($j = 1, \dots, n$). Since f is surjective, we may choose $r_j, x_j \in A$, $\|r_j\|_A < \varepsilon$, and $a_{n+1} \in A$ such that $f(r_j) = y_j := f(a_j) - b_j - k_j b_{n+1}$, $f(x_j) = k_j$, ($j = 1, \dots, n$), and $f(a_{n+1}) = b_{n+1}$. Then

$$(a_1 - r_1, \dots, a_n - r_n) \in U_n(A)$$

and so

$$(a'_1, \dots, a'_{n+1}) := (a_1 - r_1 - x_1 a_{n+1}, \dots, a_n - r_n - x_n a_{n+1}, a_{n+1}) \in U_{n+1}(A).$$

Moreover, $f(a'_j) = f(a_j) - f(x_j)f(a_{n+1}) - f(r_j) = f(a_j) - k_j b_{n+1} - y_j = b_j$ for $1 \leq j \leq n$ and $f(a'_{n+1}) = b_{n+1}$. Hence $\underline{f}(U_{n+1}(A)) = U_{n+1}(B)$.

Next, we present relations between these notions of stable rank. Most of these relations are known and can be found in the papers of Corach and Larotonda, see [3], and Suárez, see [23, 24]. Many of the proofs in these papers are based on far reaching concepts and techniques from algebraic geometry, such as Serre fibrations and homotopy classes. For the reader's convenience we present short direct proofs of some of these facts.

Proposition 2.1. [3, p. 293] *Suppose that $U_n(A)$ is dense in A^n . Then the stable rank of A is less than n . Namely, $\text{bsr}(A) \leq \text{tsr}(A)$.*

Proof. Let $(f_1, \dots, f_n, h) \in U_{n+1}(A)$. Then there exist $x_j \in A$ and $x \in A$ so that $1 = \sum_{j=1}^n x_j f_j + xh$. Since $U_n(A)$ is dense in A^n , for every $\varepsilon > 0$, there exists $(u_1, \dots, u_n) \in U_n(A)$ so that $\|u_j - x_j\|_A < \varepsilon$. Also $x = \sum_{j=1}^n h_j u_j$ for some $h_j \in A$ because (u_1, \dots, u_n) is invertible. Hence

$$\begin{aligned} \sum_{j=1}^n u_j(f_j + h_j h) &= \sum_{j=1}^n u_j f_j + xh \\ &= \left(\sum_{j=1}^n x_j f_j + xh \right) + \sum_{j=1}^n (u_j - x_j) f_j = 1 + u, \end{aligned}$$

where we have defined $u := \sum_{j=1}^n (u_j - x_j)f_j$. Moreover, we have $\|u\|_A \leq \varepsilon \sum_{j=1}^n \|f_j\|_A$. Hence for $\varepsilon > 0$ small enough, $1 + u$ is invertible in A , and so $(f_1 + h_1h, \dots, f_n + h_nh) \in U_n(A)$. \square

The following lemma is due to Corach and Suárez [4, 5].

Lemma 2.2. [4, p. 636] and [5, p. 608] *Let A be a commutative unital Banach algebra. Then, for $g \in A$, the set*

$$R_n(g) = \{(f_1, \dots, f_n) \in A^n : (f_1, \dots, f_n, g) \text{ is reducible}\}$$

is open-closed inside

$$I_n(g) = \{(f_1, \dots, f_n) \in A^n : (f_1, \dots, f_n, g) \in U_{n+1}(A)\}.$$

In particular, for $n = 1$, if $\varphi : [0, 1] \rightarrow I_1(g)$ is a continuous curve and $(\varphi(0), g)$ is reducible, then $(\varphi(1), g)$ is reducible.

A very useful characterization of the Bass stable rank is the following. Here the equivalence of items (2) and (3) was known (see [3]).

Theorem 2.3. *Let A be a commutative unital Banach algebra. The following assertions are equivalent:*

- (1) $\pi(U_n(A))$ is dense in $U_n(A/I)$ for every closed ideal I in A ;
- (2) $\text{bsr}(A) \leq n$;
- (3) $\pi(U_n(A)) = U_n(A/I)$ for every closed ideal I in A .

Here $\pi : A \rightarrow A/I$ is the canonical quotient mapping and π the associated map on A^n .

Proof. (1) \implies (2): Let $(a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(A)$. Consider the closure, I , of the ideal generated by a_{n+1} . Then A/I is a Banach algebra under the quotient norm and $(a_1 + I, \dots, a_n + I) \in U_n(A/I)$. By (1), there exists a sequence $(b_1^{(k)}, \dots, b_n^{(k)}) \in U_n(A)$ such that $\|\pi(b_j^{(k)}) - \pi(a_j)\|_{A/I} \rightarrow 0$ as $k \rightarrow \infty$ for $j = 1, \dots, n$. Hence there are $x_j^{(k)} \in A$ so that

$$\|a_j - b_j^{(k)} + x_j^{(k)}a_{n+1}\|_A \rightarrow 0.$$

Now for every k we have that the $n + 1$ -tuples

$$(b_1^{(k)} - x_1^{(k)}a_{n+1}, \dots, b_n^{(k)} - x_n^{(k)}a_{n+1}, a_{n+1})$$

are invertible and reducible, since $(b_j^{(k)} - x_j^{(k)}a_{n+1}) + x_j^{(k)}a_{n+1} = b_j^{(k)}$ and $(b_1^{(k)}, \dots, b_n^{(k)}) \in U_n(A)$. Using Lemma 2.2, which tells us that $R_n(a_{n+1})$ is closed inside $I_n(a_{n+1})$, and noticing that $b_j^{(k)} - x_j^{(k)}a_{n+1} \rightarrow a_j$ for $j = 1, \dots, n$, we see that $(a_1, \dots, a_n, a_{n+1})$ is reducible and so $\text{bsr}(A) \leq n$.

(2) \implies (3): This appears in [3, p. 296]. For the reader's convenience we present the argument. Let $(a_1 + I, \dots, a_n + I) \in U_n(A/I)$. Then there exist $y_1, \dots, y_n \in A$ and $b \in I$ such that $\sum_{j=1}^n y_j a_j = 1 + b$. Hence $(a_1, \dots, a_n, b) \in U_{n+1}(A)$ and so, by (2), there exists $x_1, \dots, x_n \in A$ such that

$$(a_1 + x_1b, \dots, a_n + x_nb) \in U_n(A).$$

It is clear that $\pi(a_j + x_j b) = a_j + I$. Hence $\underline{\pi}(U_n(A)) = U_n(A/I)$.

(3) \implies (1): This is immediate. \square

Parts of the following result appear without proof in [6, p. 542].

Theorem 2.4. *If A is a commutative unital Banach algebra, then*

$$\text{bsr}(A) = \text{ssr}(A) \leq \text{dsr}(A) \leq \text{tsr}(A).$$

Proof. The assertion that $\text{bsr}(A) = \text{ssr}(A)$ follows from Theorem 2.3. Indeed, let $n = \text{ssr}(A) < \infty$. Since for any closed ideal I the canonical map $\pi : A \rightarrow A/I$ is surjective, $\text{ssr}(A) = n$ implies that $\underline{\pi}(U_n(A)) = U_n(A/I)$. Hence, by Theorem 2.3, $m := \text{bsr}(A) \leq n$. To show that $n \leq m$, let $f : A \rightarrow B$ be a surjective homomorphism. Then the canonical injection $\check{f} : \check{A} = A/\text{Ker } f \hookrightarrow B$ is an algebra isomorphism and so $U_m(\check{A})$ is mapped onto $U_m(B)$ by \check{f} . Since $m = \text{bsr}(A)$, by Theorem 2.3, $\underline{\pi}(U_m(A)) = U_m(A/\text{Ker } f)$. Thus $\underline{f}(U_m(A)) = U_m(B)$. This means that $\text{ssr}(A) \leq m$. All together we have shown that $\text{bsr}(A) = \text{ssr}(A)$.

Now suppose that $\text{dsr}(A) = n$. Consider the algebra $B := A/I$, where I is any closed ideal in A . Then the assertion that $\text{bsr}(A) \leq \text{dsr}(A)$ follows from Theorem 2.3 when applied to the epimorphism $f = \pi$.

Next we suppose that $\text{tsr}(A) = n$. To show that $\text{dsr}(A) \leq \text{tsr}(A)$, we note that if $f : A \rightarrow B$ has dense image, then $\underline{f} : A^n \rightarrow B^n$ has dense image as well. Now, the continuity of f and the density of $U_n(A)$ in A^n imply that

$$\overline{\underline{f}(U_n(A))} \supseteq \underline{f}(\overline{U_n(A)}) = \underline{f}(A^n).$$

Therefore,

$$U_n(B) \subseteq B^n = \overline{\underline{f}(A^n)} \subseteq \overline{\underline{f}(U_n(A))}.$$

Since $\underline{f}(U_n(A)) \subseteq U_n(B)$, we finally obtain that $\underline{f}(U_n(A))$ is dense in $U_n(B)$. Therefore $\text{dsr}(A) \leq n = \text{tsr}(A)$. \square

It is not known whether always $\text{bsr}(A) = \text{dsr}(A)$. For $A = H^\infty$, for instance, we have $\text{bsr}(H^\infty) = \text{dsr}(H^\infty) = 1$ (see [13, 23, 26]) and $\text{tsr}(H^\infty) = 2$ (see [24]); for $A = H_{\mathbb{R}}^\infty = \{f \in H^\infty : \overline{f(\bar{z})} = f(z)\}$ we have $\text{bsr}(H_{\mathbb{R}}^\infty) = \text{dsr}(H_{\mathbb{R}}^\infty) = \text{tsr}(H_{\mathbb{R}}^\infty) = 2$, (see [12]).

Our next result, which seems to be new, gives a version of Lemma 2.2 with bounds.

Proposition 2.5. *Let (f, g) be an invertible pair in the commutative unital Banach algebra A . Suppose that f_n converges to f and that there exist a constant $K \geq 1$ and $u_n \in A$ such that $f_n + u_n g$ is invertible with*

$$\|u_n\|_A + \|f_n + u_n g\|_A + \|(f_n + u_n g)^{-1}\|_A \leq K.$$

Then there is $h \in A$ such that $f + hg$ is invertible and

$$\|h\|_A + \|f + hg\|_A + \|(f + hg)^{-1}\|_A \leq 8K.$$

Proof. Let $(x, y) \in A^2$ be so that $1 = xf + yg$. Then

$$f_n + u_n g = f(xf_n + yg) + (y(f_n - f) + u_n)g.$$

Since $\|f_n - f\|_A \rightarrow 0$ we may choose n_0 so big that for all $n \geq n_0$ the elements $xf_n + yg$ are invertible in A , that

$$\|xf_n + yg\|_A \leq 2 \text{ and } \|(xf_n + yg)^{-1}\|_A \leq 2,$$

and so that

$$\|f - f_n\|_A \leq \min\{1, \|y\|_A^{-1}\}.$$

If we let

$$h = (y(f_n - f) + u_n)(xf_n + yg)^{-1},$$

then $f + hg$ is invertible. Noticing that by hypothesis $\|u_n\|_A \leq K$, we get

$$\|h\|_A \leq 2(1 + K),$$

$$\|(f_n + u_n g)(xf_n + yg)^{-1}\|_A \leq 2K$$

and

$$\|(xf_n + yg)(f_n + u_n g)^{-1}\|_A \leq 2K.$$

Since $K \geq 1$,

$$\|h\|_A + \|f + hg\|_A + \|(f + hg)^{-1}\|_A \leq 8K.$$

□

3. THE STABLE RANKS OF $H^\infty + C$

It is the aim of this section to determine the stable ranks defined above for the Sarason algebra $H^\infty + C$. To this end let us recall the following Theorem of S. Treil [26] that tells us in particular that $\text{bsr}(H^\infty) = 1$.

Theorem 3.1 (Treil). *There exists a constant $C(\delta)$ depending only on $\delta \in]0, 1[$ such that for every pair (f, g) of elements in the unit ball of H^∞ satisfying $|f| + |g| \geq \delta > 0$ in \mathbb{D} , there are functions $u, h \in H^\infty$ with u invertible in H^∞ , such that $1 = uf + hg$ and $\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta)$.*

The following well known result can easily be deduced from Treil's Theorem.

Proposition 3.2. *There exists a constant $C(\delta)$ depending only on δ such that for every f_1, \dots, f_n in H^∞ satisfying $1 \geq \sum_{j=1}^n |f_j| \geq \delta > 0$ in \mathbb{D} , there exist $a_j, t_j \in H^\infty$ bounded by $C(\delta)$ so that*

$$1 = \sum_{j=1}^{n-1} a_j(f_j + t_j f_n).$$

Proof. By the H^∞ -Corona Theorem, [2], there is a constant $C_1(\delta)$ such that the Bezout equation $\sum_{j=1}^n x_j f_j = 1$ admits a solution $(x_1, \dots, x_n) \in (H^\infty)^n$ with

$$\sum_{j=1}^n \|x_j\|_\infty \leq C_1(\delta).$$

We may assume that $C_1(\delta) \geq 1$. Now

$$1 \leq \|x_1\|_\infty |f_1| + \left| \sum_{j=2}^n x_j f_j \right| \leq C_1(\delta) \left(|f_1| + \left| \sum_{j=2}^n x_j f_j \right| \right);$$

hence

$$2 + C_1(\delta) \geq |f_1| + \left| \sum_{j=2}^n x_j f_j \right| \geq \frac{1}{C_1(\delta)} := \varepsilon > 0.$$

By Treil's Theorem there is a constant $C_2(\varepsilon)$ and $u, v \in H^\infty$ such that u is invertible in H^∞ , $\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \leq C_2(\varepsilon)$ with

$$1 = u f_1 + v \left(\sum_{j=2}^n x_j f_j \right).$$

The latter equation can be re-written as

$$1 = u(f_1 + u^{-1} v x_n f_n) + \sum_{j=2}^{n-1} v x_j (f_j + 0 \cdot f_n).$$

It is clear that the functions $a_1 := u$, $t_1 := u^{-1} v x_n$, $a_j := v x_j$ and $t_j := 0$ for $j = 2, \dots, n-1$ are bounded by a constant $C(\delta)$ depending only on δ . \square

The following Theorem given by Laroco [11, p. 819] will be essential for our determination of the stable rank of $H^\infty + C$.

Theorem 3.3 (Laroco). *Let $f \in H^\infty$. Then, for every $\varepsilon > 0$, there exist a Blaschke product B and an outer function, v , invertible in H^∞ such that*

$$\|f - Bv\|_\infty < \varepsilon \text{ and } |v| \geq \varepsilon/4 \text{ on } \partial\mathbb{D},$$

as well as $\|v\|_\infty \leq 1 + \|f\|_\infty$.

We note that, due to the fact that v is invertible, we actually have $|v| \geq \varepsilon/4$ on \mathbb{D} . We additionally need the following lemma.

Lemma 3.4. *Let B be a Blaschke product and $g \in H^\infty$ with norm less than or equal to one. Suppose that $|B| + |g| \geq \delta > 0$ on $M(H^\infty + C)$. Then there exists a constant $C(\delta)$, depending only on δ , and functions h and u in $H^\infty + C$ with u invertible in $H^\infty + C$ so that $\|u\|_\infty, \|u^{-1}\|_\infty$ and $\|h\|_\infty$ are bounded by $C(\delta)$ and such that*

$$1 = uB + hg.$$

Proof. By continuity, there exists $r > 0$ such that on $\{r \leq |z| < 1\}$

$$|B| + |g| \geq \delta/2.$$

Let B^* be a tail of B , so that $|B^*| \geq \delta/2$ on $\{|z| \leq r\}$. Hence

$$|B^*| + |g| \geq \delta/2 \text{ on } M(H^\infty).$$

Let $b = B/B^*$. Note that b is a finite Blaschke product and hence $\bar{b} \in H^\infty + C$. By Treil's result [26] (here Theorem 3.1), there is a constant $C(\delta)$ and two functions $R, h \in H^\infty$, R invertible in H^∞ , such that

$$1 = RB^* + hg$$

and

$$\|R\|_\infty + \|R^{-1}\|_\infty + \|h\|_\infty \leq C(\delta/2).$$

Therefore, as $H^\infty + C$ -functions:

$$1 = (R\bar{b})B + hg.$$

□

Theorem 3.5. *The Bass stable rank of $H^\infty + C$ equals one.*

Proof. Let (φ, ψ) be an invertible pair in $H^\infty + C$. We may assume that $\|\varphi\|_\infty \leq 1$ and $\|\psi\|_\infty \leq 1$. Since $H^\infty + C$ is the uniform closure of the set of functions $\{\bar{z}^n f : f \in H^\infty, n \in \mathbb{N}\}$ (see [7]), there exists $n \in \mathbb{N}$ and $f \in H^\infty$ such that $\|\varphi - f\bar{z}^n\|_\infty < \varepsilon$. By Theorem 3.3, there is a Blaschke product B and a function v invertible in H^∞ such that $\|f - vB\|_\infty < \varepsilon$. Hence the set of functions

$$\{\bar{z}^n vB : n \in \mathbb{N}, B \text{ Blaschke}, v \text{ invertible in } H^\infty\}$$

is dense in $H^\infty + C$. By Lemma 2.2 and the fact that the factors $\bar{z}^n v$ are invertible in $H^\infty + C$, it suffices to show the reducibility of the pairs (B, ψ) , where B is any Blaschke product such that $|B| + |\psi| \geq \delta > 0$ on $M(H^\infty + C)$.

To do this, we shall use Lemma 3.4. Choose $n \in \mathbb{N}$ and $g \in H^\infty$, $\|g\|_\infty \leq 1$, so that

$$\|\bar{z}^n g - \psi\|_\infty < \min\left\{\frac{\delta}{2}, \frac{1}{2C(\delta/2)}\right\},$$

where $C(\delta)$ is the constant from Lemma 3.4. Now consider the pair (B, g) . We obviously have (on $M(H^\infty + C)$)

$$|B| + |g| = |B| + |\bar{z}^n g| \geq |B| + |\psi| - |\psi - \bar{z}^n g| \geq \delta/2.$$

By Lemma 3.4, there exists $u \in H^\infty + C$, u invertible, and $h \in H^\infty + C$ with $\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta/2)$ such that

$$1 = uB + hg.$$

Hence

$$\begin{aligned} |uB + (hz^n)\psi| &= |uB + hg + h(z^n\psi - g)| \geq \\ 1 - \|h\|_\infty \|\psi - \bar{z}^n g\|_\infty &\geq 1 - \|h\|_\infty \frac{1}{2C(\delta/2)} \geq 1/2. \end{aligned}$$

Thus $uB + (hz^n)\psi$ is invertible in $H^\infty + C$. Hence $1 = xB + y\psi$ where $x \in H^\infty + C$ is invertible and

$$\max\{\|x\|_\infty, \|y\|_\infty\} \leq 2C(\delta/2) \text{ and } \|x^{-1}\|_\infty \leq 2C^2(\delta/2).$$

This shows that the pair (B, ψ) is reducible in $H^\infty + C$. \square

Combining Proposition 2.5 with the proof above, we get the following extension of Theorem 3.5.

Theorem 3.6. *There exists a constant $C(\delta)$ depending only on $\delta \in]0, 1[$ such that for every pair (φ, ψ) of functions in the unit ball of $H^\infty + C$ satisfying $|\varphi| + |\psi| \geq \delta$ on $M(H^\infty + C)$, there is a solution $(u, v) \in (H^\infty + C)^2$ of the Bezout equation $u\varphi + v\psi = 1$, where u is invertible in $H^\infty + C$, and such that $\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \leq C(\delta)$.*

Proof. According to Theorem 1.1, respectively Corollary 1.2, let (x, y) be a solution in $H^\infty + C$ of the Bezout equation $x\varphi + y\psi = 1$ with $\|x\|_\infty + \|y\|_\infty \leq \tilde{\chi}(\delta)$. Using Theorem 3.3, we may choose a Blaschke product B and a function h invertible in H^∞ such that

$$\|\varphi - \bar{z}^\nu B h\|_\infty < \sigma(\delta) := \min\{\delta/2, (2\tilde{\chi}(\delta))^{-1}\}$$

and $2 \geq |h| > \sigma(\delta)/4$. Since $|B| + |\psi| \geq \delta/4$ on $M(H^\infty + C)$, there exists by the proof of Theorem 3.5 above a constant $C_1(\delta)$ and a function $q \in H^\infty + C$ such that $F := B + q\psi$ is invertible in $H^\infty + C$ and so that

$$\|q\|_\infty + \|F\|_\infty + \|F^{-1}\|_\infty \leq C_1(\delta).$$

Let $f^* := \bar{z}^\nu h B$ and $u^* := \bar{z}^\nu h q$. Then $v^* := f^* + u^*\psi$ is invertible in $H^\infty + C$ and

$$\|u^*\|_\infty + \|v^*\|_\infty + \|(v^*)^{-1}\|_\infty \leq C_2(\delta) := 6C_1(\delta) + \frac{4}{\sigma(\delta)}C_1(\delta).$$

Now, as in the proof of Proposition 2.5, we see from

$$f^* + u^*\psi = \varphi(xf^* + y\psi) + (y(f^* - \varphi) + u^*)\psi$$

that $1 = u\varphi + v\psi$ with

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|v\|_\infty \leq C_3(\delta).$$

\square

In [24], Suárez showed that $\text{tsr}(H^\infty) = 2$. Using this result we deduce the topological stable rank for $H^\infty + C$.

Theorem 3.7. *The topological stable rank of $H^\infty + C$ is 2.*

Proof. First we show that the topological stable rank of $H^\infty + C$ is at most 2. Let $(\varphi_1, \varphi_2) \in (H^\infty + C)^2$. Approximate φ_j by functions of the form $\frac{z^{n_j}}{z^{n_j}} f_j$, where $f_j \in H^\infty$, say $\|\frac{z^{n_j}}{z^{n_j}} f_j - \varphi_j\|_\infty < \varepsilon$, $j = 1, 2$.

Since the topological stable rank of H^∞ is two, there exists $g_j \in H^\infty$ such that $\|g_j - f_j\|_\infty \leq \varepsilon$ and so that $(g_1, g_2) \in U_2(H^\infty)$. Obviously

$(\overline{z^{n_1}}g_1, \overline{z^{n_2}}g_2) \in U_2(H^\infty + C)$ and $\|\overline{z^{n_j}}g_j - \varphi_j\|_\infty < 2\varepsilon$. Thus $\text{tsr}(H^\infty + C) \leq 2$.

Let b be an (infinite) interpolating Blaschke product. Note that b is not invertible in $H^\infty + C$. Let $m \in M(H^\infty + C)$ be a zero of b and let L_m be the associated Hoffman map. Since $b \circ L_m$ is analytic, but not identically zero, we see that b cannot be uniformly approximated on $M(H^\infty + C)$ by invertibles in $H^\infty + C$, say u_n , since otherwise $\|b \circ L_m - u_n \circ L_m\|_\infty \rightarrow 0$; a contradiction to Rouché's theorem. Thus $\text{tsr}(H^\infty + C) \geq 2$. \square

Next we deal with the dense stable rank of $H^\infty + C$. Recall that if A is a commutative unital Banach algebra, then E is an A -convex subset of $M(A)$ if

$$\forall x \notin E \exists f \in A : |f(x)| > \sup_E |f|.$$

We let \hat{E} denote the A -convex hull of a closed set $E \subseteq M(A)$. This is given by

$$\hat{E} = \{m \in M(A) : |f(m)| \leq \sup_E |f|, \forall f \in A\}.$$

Note that E is A -convex if and only if $E = \hat{E}$. We say that E is a proper A -convex set if $E = \hat{E}$ and $\hat{E} \neq M(A)$.

Theorem 3.8. *The dense stable rank of $H^\infty + C$ is one.*

Proof. Let E be an $H^\infty + C$ -convex subset of $M(H^\infty + C)$. By [13] it is sufficient to prove that any function $\varphi \in H^\infty + C$ that does not vanish on E can be uniformly approximated on E by an invertible function in $H^\infty + C$. To this end, we first approximate φ on \mathbb{T} (hence on $M(H^\infty + C)$) by a function of the form $\overline{z^n}f$, where $n \in \mathbb{N}$ and $f \in H^\infty$, say $\|\varphi - \overline{z^n}f\|_\infty < \varepsilon/2$. By choosing ε sufficiently small, we see that f does not vanish on E , too. Let

$$\check{E} = \{m \in M(H^\infty) : |h(m)| \leq \sup_E |h|, \forall h \in H^\infty\}$$

be the H^∞ -convex hull of E . We show that $\check{E} \cap M(H^\infty + C) = E$. To this end, let $m_0 \in M(H^\infty + C) \setminus E$. Since E is $H^\infty + C$ -convex, there exists a function $\psi \in H^\infty + C$ such that $|\psi(m_0)| > \sup_E |\psi|$. Uniformly approximating ψ by a function of the form $\overline{z^n}g$, shows that $|g(m_0)| > \sup_E |g|$ for some $g \in H^\infty$. Thus $m_0 \notin \check{E}$, the H^∞ -convex hull of E . This shows that $\check{E} \cap M(H^\infty + C) = E$.

Now \check{E} can be written as $\check{E} = E \cup S$, where $S = \check{E} \cap \mathbb{D}$. We probably have $S = \emptyset$, however this isn't necessary for the rest of the proof. Note that \check{E} is closed and $\overline{S} \setminus \mathbb{D} \subseteq E$. Recall that our function f above does not vanish on E ; hence f can have only finitely many zeros in S . Write $f = qF$, where q is the finite Blaschke product formed with the zeros of f in S . Thus F does not vanish on \hat{E} . By [13], for every $\varepsilon > 0$, there is an invertible function $h \in H^\infty$ such that $\sup_{\hat{E}} |F - h| < \varepsilon/2$. Hence, by noticing that $|q| = 1$ on $M(H^\infty + C)$, we have $|qh - f| = |qh - qF| < \varepsilon/2$ on E . Therefore, on E ,

$$\|\overline{z^n}qh - \varphi\|_\infty \leq \|\overline{z^n}(qh - f)\|_\infty + \|\varphi - \overline{z^n}f\|_\infty < \varepsilon.$$

Since $\overline{z}^n qh$ does not vanish on $M(H^\infty + C)$, that function is invertible in $H^\infty + C$. \square

Remark Using Theorem 2.4 and Theorem 3.8 we get a second proof of the fact that $\text{bsr}(H^\infty + C) = 1$ (see Theorem 3.5).

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DÉPARTEMENT DE MATHÉMATIQUES, LMAM, UMR 7122, UNIVERSITÉ PAUL VERLAINE, ÎLE DU SAULCY, F-57045 METZ, FRANCE

E-mail address: `mortini@poncelet.univ-metz.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, LeCONTE COLLEGE, 1523 GREENE STREET, COLUMBIA, SC USA 29206

E-mail address: `wick@math.sc.edu`